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# NON-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS AND DIFFERENCE EQUATIONS

CONTINUATION

Gregory T. McAllister

RDT & E Project No. 1M010501A003

# BALLISTIC RESEARCH LABORATORIES

# ABERDEEN PROVING GROUND, MARYLAND

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Computing Laboratory

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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Gregory T. McAllister/ilm  
Aberdeen Proving Ground, Md.  
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NON-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS AND  
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ABSTRACT

The Dirichlet problem for the non-linear elliptic partial differential equation

$$a(x,y,u(x,y))u_{xx} + c(x,y,u(x,y))u_{yy} - \gamma(x,y,u(x,y))u = 0$$

is studied. It is assumed that the coefficients are strictly positive and Lipschitz in the argument  $u(x,y)$ . It is then proved that the solution may be uniformly approximated by the solution to the associated difference equation provided that a certain inequality, relating bounds on the coefficients, is satisfied.

This paper studies a difference method for a non-linear elliptic partial differential equation of the form

$$a(x,y,u(x,y))u_{xx} + c(x,y,u(x,y))u_{yy} - \gamma(x,y,u(x,y))u = 0 \quad (1)$$

where  $(x,y)$  belongs to the open unit square  $\Omega$ ,  $u(x,y) = \phi(x,y)$  for  $(x,y)$  on the boundary,  $\partial\Omega$ , of  $\Omega$  and  $a$ ,  $c$ ,  $\gamma$  are Lipschitz functions of the arguments  $x$ ,  $y$ , and  $u(x,y)$ . The constant  $K_1$  will denote the maximum of the Lipschitz constants associated with  $a$ ,  $c$  and  $\gamma$ .

We shall assume that there exist positive constants  $K_0$  and  $m$  such that

$$K_0 \geq a(x,y,u(x,y)) \geq m > 0,$$

$$K_0 \geq c(x,y,u(x,y)) \geq m > 0,$$

and

$$K_0 \geq \gamma(x,y,u(x,y)) \geq m > 0.$$

Place a square grid on  $\Omega$ . The lattice points are expressible in the form  $(mh, nh)$  with  $m$  and  $n$  running through a set of integers and with  $h$  denoting the mesh size. Let  $P_i = (x,y)$  be a lattice point. The neighbors of  $P_i$ , denoted by  $P_{iv}$  ( $v = 1, \dots, 4$ ), are the points of the form  $(x+h,y)$ ,  $(x-h,y)$ ,  $(x,y+h)$ ,  $(x,y-h)$ . Let  $\Omega_h$  consist of those lattice points for which all of their neighbors are in  $\Omega$ . Let  $\partial\Omega_h$  denote the boundary of  $\Omega_h$ . We place a lexicographic order on the points of  $\Omega_h$  and denote these points by  $P_1, \dots, P_N$ . The points of  $\partial\Omega_h$  are denoted by  $P_{N+1}, \dots, P_M$ . For a function  $f(x,y)$ , defined on  $\Omega_h$ , we denote by  $f_i$  the value of  $f$  at  $P_i$ . For any  $M$ -dimensional vector  $\xi$ , the quantity  $||\xi||$  denotes the  $\max_i |\xi_i|$  or  $\max_{i,j} |\xi_{ij}|$  for  $\xi$  a vector or a matrix.

In §1 a maximum principle is proved following the methods in Bers<sup>\*</sup> [1]. Existence and uniqueness of a solution of the difference equation associated with (1) is proved in §2 under a certain assumption governing the domain of dependence of the coefficients  $a$ ,  $b$  and  $\gamma$  upon the solution function  $u(x,y)$ . In §3 convergence of the solution of the difference equation to the solution of the differential equation is proved. The condition imposed on the coefficients, concerning the domain of dependence, is removed in §4 by defining a new sequence of problems which satisfy the condition and which converge to the original problem.

<sup>\*</sup> He applies these methods to equations whose leading coefficient does not depend on the solution.

### §1. A Maximum Principle

Let  $\xi = (\xi_k)$  be any M-component vector. A difference operator over  $\Omega_h$  corresponding to (1), is ([2], p. 190)

$$L_h[\xi_i; U_i] = h^{-2} \left\{ -2(a_i + c_i)U_i + a_i(U_{i1} + U_{i3}) + c_i(U_{i2} + U_{i4}) \right\} - \gamma_i U_i \quad (1.1)$$

where the arguments of the coefficients  $a_i, c_i$ , and  $\gamma_i$  are  $(x_i, y_i, \xi_i)$ .

Let the matrix  $L(\xi) = (L^{ik}(\xi))$  be defined such that the  $(i,k)$ -th entry is the coefficient of  $U_k$  when  $L_h[\xi_i; U_i]$  is evaluated at the  $i$ -th point of the mesh. The matrix  $-L(\xi)$  is monotone ([3], p. 45) and hence its determinant,  $\det(L(\xi))$ , is non-zero.

We now state the maximum principle.

Theorem 1: Let  $U = (U_1, \dots, U_N, U_{N+1}, \dots, U_M)$  be any M-component vector. Then

$$\max_i |U_i| \leq C_1 \max_i |L_h[\xi_i; U_i]| + \max_j |U_j| \quad (1.2)$$

where  $i = 1, \dots, N$ ,  $j = N+1, \dots, M$ , and  $C_1$  is a constant independent of  $\xi$  and  $U$ .

Proof: Consider the case where, for all  $i$ ,

$$\begin{aligned} L_h[\xi_i; U_i] &= 0. \\ \text{If } B_i &= 2(a_i + c_i) + h^2 \gamma_i, \text{ then } U_i = \sum_{\nu=1}^4 \frac{\lambda_{i\nu}}{B_i} U_{i\nu} \end{aligned} \quad (1.3)$$

where

$$\lambda_{i2} = \lambda_{i4} = c_i,$$

$$\lambda_{i1} = \lambda_{i3} = a_i.$$

Since  $\lambda_{i\nu} > 0$  ( $\nu=1, \dots, 4$ ) and  $0 < \sum_{\nu=1}^4 \frac{\lambda_{i\nu}}{B_i} < 1$ ,

(1.3) implies that

$$U_i \begin{cases} \leq \max(U_{i1}, \dots, U_{i4}) \text{ for } U_i \geq 0 \\ \leq \min(U_{i1}, \dots, U_{i4}) \text{ for } U_i \leq 0 \end{cases} \quad (1.4)$$

The equality in (1.4) holds only if  $U_{i\ell} = U_{i4}$ , for  $\ell = 1, 2, 3$ , and  $\gamma_i U_i = 0$ .

Hence,

$$\min_{\partial\Omega_h} (0, \min U) \leq U_i \leq \max_{\partial\Omega_h} (0, \max U) \quad (1.5)$$

for  $i = 1, \dots, N$ .

Consider the case,

$$\begin{aligned} L_h[\xi_i; U_i] &= k_i & i &= 1, \dots, N. \\ U_j &= v_j & j &= N+1, \dots, M. \end{aligned} \quad (1.6)$$

Using the definition of  $L$  we have that

$$\sum_{s=1}^N L^{is}(\xi) U_s = k_i + \sum_{j=N+1}^M \sigma^{ij}(\xi) v_j \quad (i = 1, \dots, N) \quad (1.7)$$

where  $\sigma(\xi) = (\sigma^{ij}(\xi))$  is defined in the following way:  $\sigma^{ij}(\xi)$  is the coefficient of the  $j$ th boundary point when we consider  $L_h[\xi_i; U_i]$ . Note that  $\sigma(\xi)$  is rectangular, singular and the arguments are the coefficients  $a(x, y, \xi)$ ,  $c(x, y, \xi)$ , and  $\gamma(x, y, \xi)$  evaluated at interior points of the mesh.

Letting  $U = (U_1, \dots, U_N)^T$ ,  $K = (k_{N+1}, \dots, k_M)^T$  and  $v = (v_{N+1}, \dots, v_M)^T$ , where the superscript  $T$  denotes the transpose, gives

$$L(\xi)U = K + \sigma(\xi)v.$$

Since  $L^{-1}(\xi)$  exists, we have

$$U = L^{-1}(\xi)K + L^{-1}(\xi)\sigma(\xi)v$$

or, for  $i = 1, \dots, N$ ,

$$U_i = \sum_{s=1}^N G^{is}(\xi)k_s + \sum_{j=N+1}^M \left( \sum_{\ell=1}^N G^{i\ell}(\xi)\sigma^{\ell j}(\xi) \right) v_j, \quad (1.8)$$

where  $G(\xi) = L^{-1}(\xi)$ .

If

$$\Gamma^{ij}(\xi) = \sum_{\ell=1}^N G^{i\ell}(\xi) \sigma^{ij}(\xi) ,$$

then

$$\max |U| \leq C_1(\xi) \max_{\Omega_h} |L_h[\xi; U]| + C_2(\xi) \max_{\partial\Omega_h} |U| \quad (1.9)$$

where

$$C_1(\xi) = \max_i \sum_{s=i}^N |G^{is}(\xi)|$$

$$C_2(\xi) = \max_i \sum_{j=N+1}^M |\Gamma^{ij}(\xi)| .$$

In order to prove that  $C_2(\xi) \leq 1$ , it will be shown that  $\Gamma^{i\ell}(\xi)$  and

$\sum_{j=N+1}^M \Gamma^{ij}(\xi)$  lie in the interval  $[0, 1]$  for any  $\xi$ .

Let  $U$  be a solution of the equation

$$L_h[\xi_i; U_i] = 0 \quad i = 1, \dots, N.$$

with

$$v_j = \delta_{j\ell} \quad (\text{Kronecker } \delta) \quad j = N+1, \dots, M.$$

Then  $U_i = \Gamma^{i\ell}(\xi)$  and by (1.5)

$$0 \leq U_i \leq 1 \quad i = 1, \dots, N.$$

Let  $U$  be a solution of the equation

$$L_h[\xi_i; U_i] = 0 \quad i = 1, \dots, N$$

with

$$v_j = 1 \quad j = N + 1, \dots, M.$$

Then

$$U_i = \sum_{j=N+1}^M \Gamma^{ij}(\xi)$$

and by (1.5)

$$0 \leq U_i \leq 1.$$

We will now obtain a bound for  $C_1(\xi)$ .

Let

$$v(x, y) = e^{\alpha x}$$

where  $\alpha = [(K_0 + 1)m^{-1}]^{1/2}$  and  $\Omega$  is assumed to lie in the strip  $0 \leq x \leq 1$ .

Then

$$0 \leq v \leq e^\alpha$$

and

$$L_h[\xi_i; v_i] \geq e^{\alpha x} [2a_i \alpha^2 - \gamma_i] \geq 1.$$

Observing that  $G^{is}(\xi) \leq 0$ , since  $-L(\xi)$  is monotone, (1.8) gives

$$\sum_{s=1}^N |G^{is}(\xi)| \cdot L_h[\xi_s; v_s] \leq \sum_{j=N+1}^M \Gamma^{ij}(\xi) v_j.$$

Hence, for all  $\xi$

$$\sum_{s=1}^N |G^{is}(\xi)| \leq \frac{\max V}{\frac{\Omega}{\min L_h[\xi; V]}} \leq e^\alpha .$$

## §2. Existence and Uniqueness

It has been shown that any solution  $U$  of the equation

$$L_h[\xi_i; U_i] = 0 \quad (2.1)$$

must satisfy the inequality

$$\max_i |U_i| \leq \max_j |v_j| \quad (2.2)$$

where  $i = 1, \dots, N$ ,  $v_j = \phi(P_j)$ , and  $j = N+1, \dots, M$ . By [1] there exists a unique solution  $U$  to (2.1) for any given  $\xi$ . Hence to every  $M$ -vector  $\xi$  there corresponds a unique solution  $U$ . The dependence of  $U$  on  $\xi$  will be denoted by

$$U = \phi(\xi) . \quad (2.3)$$

The relation  $\phi$  is a function satisfying (2.2). By (1.8) the function  $\phi$  is continuous and, by Theorem 1, can be restricted to the  $N$ -dimensional sphere of radius  $\max_j |v_j|$ .

We are seeking a fixed-point solution of (2.1) i.e. solutions  $U$  for which  $U = \phi(U)$ . Application of Brouwer's Fixed-point Theorem ([4], p. 357) assures the existence of at least one such fixed-point of (2.3).

Before proceeding with a uniqueness proof the following assumption concerning our differential equation will be made: Condition A. There exists a set  $\Omega'$  such that  $\Omega'$  and its closure are proper subsets of  $\Omega$  and in  $\Omega - \Omega'$  the coefficients  $a$ ,  $b$  and  $\gamma$  do not depend on the function  $u$ .

Theorem 2: Let  $a(x, y, u)$ ,  $b(x, y, u)$  and  $\gamma(x, y, u)$  be Lipschitz functions of their arguments  $(x, y, u)$ . Let  $h$  be so small that  $K_2$  satisfies  $50K_2^{-2} < (5N^4 C_0^2)^{-1/N}$  where  $K_2 = \max \{K_0, K_1\}$ , and  $C_0 = \max \{||\sigma||, ||v||\}$ . Then there exists at most one fixed-point solution.

Proof: Assume there are two fixed-point solutions, say  $V$  and  $W$ . Then (1.8) gives

$$|v_i - w_i| = |\Phi(v) - \Phi(w)|$$

$$\begin{aligned} &= \left| \sum_{j=N+1}^M \sum_{\ell=1}^N (G^{i\ell}(v)\sigma^{\ell j}(v) - G^{i\ell}(w)\sigma^{\ell j}(w))v_j \right| \quad (2.4) \\ &\leq \|\sigma\| \cdot \|v\| \sum_{j=N+1}^M \sum_{\ell=1}^N |G^{i\ell}(v) - G^{i\ell}(w)|, \end{aligned}$$

where the last inequality follows in virtue of Condition A.

Since  $G$  is the inverse of the matrix  $L$ , one has

$$G^{is}(v) = L^{si}(v) \cdot [\det[L(v)]]^{-1}$$

where  $L^{si}(v)$  is the  $(s,i)$  - cofactor of  $L(v)$ .

Then

$$\begin{aligned} |G^{is}(v) - G^{is}(w)| &= \left| \frac{\det[L(w)] L^{si}(v) - \det[L(v)] \cdot L^{si}(w)}{\det[L(v)] \cdot \det[L(w)]} \right| \\ &\leq m^{-2N} \cdot (10K_2)^N \{ |L^{si}(v) - L^{si}(w)| + |\det[L(v)] - \det[L(w)]| \}, \end{aligned}$$

by virtue of ([5], p. 70)

$$|\det[L(\xi)]| \geq \prod_{i=1}^N \gamma_i(\xi) \geq m^N$$

and ([5], p. 70)

$$|\det[L(\xi)]| \leq (10K_2)^N$$

for any  $\xi$ . Now  $L^{si}(\xi)$  consists of sums and sums of products of Lipschitz functions. Hence,

$$|L^{si}(v) - L^{si}(w)| \leq (N-1)5^{N-1}K_2^{N-1} |v - w| . \quad (2.6)$$

Similarly,

$$|\det[L(V)] - \det[L(W)]| \leq N^N K_2^N |V - W| . \quad (2.7)$$

Substituting (2.5), (2.6) and (2.7) into (2.4) yields

$$|V_i - W_i| \leq (M - N - 1) N^3 C_0^2 (50 K_2^M)^{-2} N |V - W| . \quad (2.8)$$

If  $N$  is chosen sufficiently large, then the coefficient of  $|V - W|$  in (2.8) remains less than one and we have the contradiction

$$|V - W| < |V - W| .$$

Remark: This proof shows that the mapping  $\phi$  is a Lipschitz function with Lipschitz constant less than 1.

### §3. Convergence

Theorem 3: If the hypotheses of Theorem 2 are satisfied, the fixed-point solution of the difference equation (1.1) converges uniformly to the solution of the partial differential equation (1).

Proof: Assume the solution  $u$  is completely known. Let  $W^{(p)}$  be the solution of the difference equation

$$a_i D_{h,xx} W_i^{(p)} + c_i D_{h,yy} W_i^{(p)} - \gamma_i W_i^{(p)} = 0 \quad (3.1)$$

with

$$a_i = a(x_i, y_i, W_i^{(p-1)}) ,$$

$$c_i = c(x_i, y_i, W_i^{(p-1)}) ,$$

$$\gamma_i = \gamma(x_i, y_i, W_i^{(p-1)}) ,$$

$$W_j^{(p)} = u_j , \quad j = N + 1, \dots, M, \text{ and}$$

$$W_i^{(0)} = u_i ;$$

here  $D_{h,xx} U_i$ , etc. are used to denote the coefficients of  $a_i$ , etc. in (1.1).

From (3.1) and (1.8) we have

$$|w_i^{(p)} - w_i^{(p-1)}| = \left| \sum_{j=N+1}^M \Gamma_{(p-1)}^{ij} v_j - \Gamma_{(p-2)}^{ij} v_j \right| \quad (3.2)$$

$$\leq C_3 \cdot ||w^{(p-1)} - w^{(p-2)}||$$

where  $C_3 = 5N^4 C_0^2 (50K_3 m_0^{-2})^N$  and  $\Gamma_{(n-1)}^{ij}$  corresponds to the  $n$ -th equation in

(3.1). From [1] we conclude that

$$||w^{(p)} - w^{(p-1)}|| \leq C_3^{p-1} \cdot \epsilon(h) \quad (3.3)$$

where  $\epsilon(h) \rightarrow 0$  as  $h \rightarrow 0$ . Hence, the sequence  $\{w^{(p)}\}$  is a Cauchy sequence and has a limit which we call  $v^{(h)}$ . This limit is a fixed-point solution of our difference equation.

Repeated application of (3.3) yields

$$||w^{(p)} - w^{(0)}|| \leq C_4(p) \cdot \epsilon(h) \quad (3.4)$$

where  $C_4(p) = \sum_{j=1}^{p-1} C_3^j$ . By hypothesis we have that  $C_4(p)$  is uniformly

bounded in  $p$  and consequently we obtain the sought inequality

$$||v^{(h)} - u|| \leq C \cdot \epsilon(h) \quad (3.5)$$

where  $C = \lim_{p \rightarrow \infty} C_4(p)$ .

#### § 4. Removal of Condition A

In §2 a condition governing the domain of non-linearity of the coefficients was introduced. Here it shall be shown how this condition may be removed.

Let  $\Omega^\epsilon = \{(1-\epsilon)x, (1-\epsilon)y: (x,y) \in \Omega\}$  with  $\epsilon > 0$ . Then  $\Omega^\epsilon$  and its closure are proper subsets of  $\Omega$ . Let  $\phi$  be extended, in a continuously differentiable way, into the exterior of  $\Omega$ .

Let a new problem be posed as follows:

Find a solution, over  $\Omega$ , of the equation

$$a^\epsilon(x, y, u)u_{xx} + c^\epsilon(x, y, u)u_{yy} - \gamma^\epsilon(x, y, u)u = 0, \quad (4.1)$$

with  $u = \phi(x, y)$  for  $(x, y) \in \partial\Omega$ . Here

$$a^\epsilon(x, y, u) = \begin{cases} a(x, y, u(\frac{x}{1-\epsilon}, \frac{y}{1-\epsilon})) & \text{for } (x, y) \in \Omega^\epsilon \\ a(x, y, \phi(\frac{x}{1-\epsilon}, \frac{y}{1-\epsilon})) & \text{for } (x, y) \in \Omega - \Omega^\epsilon \end{cases}$$

and  $c^\epsilon, \gamma^\epsilon$  are defined analogously.

Clearly  $a^\epsilon, c^\epsilon$ , and  $\gamma^\epsilon$  are Lipschitz functions of their arguments  $(x, y, u)$  and as  $\epsilon \rightarrow 0$  these coefficients converge uniformly to  $a(x, y, u)$ ,  $c(x, y, u)$  and  $\gamma(x, y, u)$ , respectively. Hence the solutions to (4.1) will converge uniformly to the solution of (1.1) as  $\epsilon \rightarrow 0$  ([6] p. 152 and [4] p. 397).

From §3 there is a uniform approximation,  $U^\epsilon$ , to the differential equation corresponding to  $u^\epsilon(x, y)$  i.e.

$$|U^\epsilon - u^\epsilon| \leq k(\epsilon, h)$$

where  $h$  is the mesh size,  $\epsilon > h$ ,  $k(\epsilon, h) \rightarrow 0$  as  $h \rightarrow 0$  and  $U^\epsilon$  is the solution of the  $\epsilon$ -difference equation. Also, for the solution  $u$  of (1.1),

$$|u^\epsilon - u| \leq \ell(\epsilon)$$

where  $\ell(\epsilon) \rightarrow 0$ . Hence,

$$|U^\epsilon - u| \leq k(\epsilon, h) + \ell(\epsilon).$$

The following is therefore proved:

Theorem 4: If the hypotheses of Theorem 2 are satisfied, if  $\epsilon > h$  and  $\epsilon \rightarrow 0$ , then there exists a solution of the difference equation which converges uniformly to the solution of the differential equation.

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 Difference equations

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 DIFFERENCE EQUATIONS  
 Gregory T. McAllister  
 BRL Report No. 1215 August 1963  
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The Dirichlet problem for the non-linear elliptic partial differential equation

$$a(x,y,u(x,y))u_{xx} + c(x,y,u(x,y))u_{yy} - \gamma(x,y,u(x,y))u = 0$$

is studied. It is assumed that the coefficients are strictly positive and Lipschitz in the argument  $u(x,y)$ . It is then proved that the solution may be uniformly approximated by the solution to the associated difference equation provided that a certain inequality, relating bounds on the coefficients, is satisfied.

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